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BOTTLENECK PROBLEMS, FUNCTIONAL EQUATIONS AND DYNAMIC PROGRAMMING

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BOTTLENECK PROBLEMS, FUNCTIONAL EQUATIONS, AND DYNAMIC PROGRAMMING

Richard Bellman

§1. Introduction.

The study of inter-industry processes is one to which a great deal of attention has been given in recent years. In particular, it has become a matter of more than theoretical interest to determine the most efficient utilization of multi-industry complexes.

In this paper of essentially expository nature we shall present a new theoretical technique, based upon the principles of the theory of dynamic programming, which may be used to treat some of the novel types of mathematical problems which arise from these studies. Subsequently, a paper will be presented containing the solutions to a number of specific problems, based upon joint work by the author and Sherman Lehman.

The problem we shall discuss is a "bottleneck problem" in the sense that the level of economic activity will be determined by the resource in shortest supply.

To begin with we shall, in the next section, present a typical problem of the bottleneck type involving the auto industry, the steel industry and the tool industry. Following this, we shall formulate the problem mathematically employing a discrete approximation. After a short discussion of the difficulties of this approach we shall turn to a continuous approximation. We will now be in a position to apply the functional equation approach ~~of~~ the theory of dynamic programming. This method used with faith and resolution will often yield the solution quite rapidly.

To prove in a simple fashion that what we have is actually a solution, we exploit the linearity of the problem and construct a dual problem. This dual problem may be used both to verify and derive a solution.

To illustrate these techniques we shall consider in detail the problem of determining the $y_1(t)$ and $y_2(t)$ which maximize $x_2(T)$ given the equations

(1) $$\frac{dx_1}{dt} = a_1 y_1(t) \quad x_1(0) = c_1 ,$$
$$\frac{dx_2}{dt} = (a_2-1)y_2(t)-y_1(t), \quad x_2(0) = c_2 ,$$

and the constraints

(2)
(a) $y_1, y_2 \geq 0$
(b) $y_1 + y_2 \leq x$
(c) $y_2 \leq b_2 x_1/a_2$
(d) $x_2 \geq 0.$

This problem is a particularly simple example of the mathematical problems which inter-industry analyses raise.

Finally, we shall briefly indicate the extension of our methods to other types of variational problems which involve non-linear functions and functionals. For those who are interested in other aspects of the theory of dynamic programming, we refer to the papers [1] - [3] listed in the bibliography.

§2. A Typical Problem.

Let us consider a model of a three-industry production process, where the individual industries are the auto industry, the steel industry, and the tool industry. These industries are to be used to produce as many autos as possible over a given time period of length T.

To simplify matters we shall assume that each industry is characterized at any time t, which to begin with we shall allow only to take the values 0,1,2,...,T, by its capacity and its stockpile. Let

(1) $x_1(t)$ = number of autos produced up to time t

$x_2(t)$ = capacity of auto factories

$x_3(t)$ = stockpile of steel

$x_4(t)$ = capacity of steel mills

$x_5(t)$ = stockpile of tools

$x_6(t)$ = capacity of tool factories

We shall assume a linear production process in the sense that output is always directly proportional to the minimum input. Thus, production is directly proportional to capacity whenever there is an abundance of raw materials and directly proportional to the quantity of raw materials allocated whenever there is an abundance of capacity. To be more precise, we postulate that

(2) a. An increase in auto, steel or tool capacity requires steel and tools

b. Production of autos requires auto capacity and steel

c. Production of steel requires only steel capacity
d. Production of tools requires tool capacity and steel.

The process proceeds in the following fashion. At the beginning of each unit time period, say t to t+1, we allocate various quantities of steel and tools, taken from their respective stockpiles, for the purposes of producing autos, steel, and tools and of increasing the auto, steel, and tool stockpiles.

Let

(3) (a) $z_i(t)$ = amount of steel allocated at time t for the purpose of increasing $x_i(t)$,

(b) $w_i(t)$ = amount of tools allocated at time t for the purpose of increasing $x_i(t)$.

Upon referring to the assumptions in (2), we see that

(4) (a) $z_3 = 0$

(b) $w_1 = w_3 = w_5 = 0$.

Combining the assumptions in (2) with those of the previous paragraph, we obtain the following equations which relate $x_i(t+1)$ to $x_i(t)$, $z_i(t)$ and $w_i(t)$:

(5)
$$\begin{aligned}
x_1(t+1) &= x_1(t) + \mathrm{Min}\,(\gamma_1 x_2(t),\ \alpha_1 z_1(t),\ \beta_1 w_1(t)) \\
x_2(t+1) &= x_2(t) + \mathrm{Min}\,(\alpha_2 z_2(t),\ \beta_2 w_2(t)) \\
x_3(t+1) &= x_3(t) - z_1(t) - z_2(t) - z_4(t) - z_5(t) - z_6(t) + \gamma_2 x_4(t) \\
x_4(t+1) &= x_4(t) + \mathrm{Min}\,(\alpha_4 z_4(t),\ \beta_4 w_4(t)) \\
x_5(t+1) &= x_5(t) - w_2(t) - w_4(t) - w_6(t) + \mathrm{Min}\,[\gamma_5 x_6(t), \alpha_5 z_5(t)] \\
x_6(t+1) &= x_6(t) + \mathrm{Min}\,(\alpha_6 z_6(t), \beta_6 w_6(t)),
\end{aligned}$$

where α_i, β_i and γ_i are constants.

The constraints upon z_i and w_i are obviously

(6) (a) $z_i, w_i \geq 0$

(b) $z_1 + z_2 + z_4 + z_5 + z_6 \leq x_3$

(c) $w_2 + w_4 + w_6 \leq x_5$,

together with the "common sense" constraints

(7) (a) $\alpha_1 z_1 = \beta_1 w_1 \leq \gamma_1 x_2$

(b) $\alpha_2 z_2 = \beta_2 w_2$

(c) $\alpha_4 z_4 = \beta_4 w_4$

(d) $\alpha_5 z_5 \leq \gamma_5 x_6$

(e) $\alpha_6 z_6 = \beta_6 w_6$

By means of these additional constraints we may eliminate w_i completely, obtaining in place of (5) the system of equations:

(8) $x_1(t+1) = x_1(t) + \alpha_1 z_1(t), \quad x_1(0) = c_1$,

$x_2(t+1) = x_2(t) + \alpha_2 z_2(t), \quad x_2(0) = c_2$,

$x_3(t+1) = x_3(t) - z_1(t) - z_2(t) - z_4(t) - z_5(t) - z_6(t) + \gamma_2 x_4(t), \quad x_3(0) = c_3$,

$x_4(t+1) = x_4(t) + \alpha_4 z_4(t), \quad x_4(0) = c_4$,

$x_5(t+1) = x_5(t) - \epsilon_2 z_2(t) - \epsilon_4 z_4(t) - \epsilon_6 z_6(t) + \alpha_5 z_5(t), \quad \epsilon_i = \alpha_i/\beta_i$,

$x_5(0) = c_5$,

$x_6(t+1) = x_6(t) + \alpha_6 z_6(t), \quad x_6(0) = c_6$.

The constraints, in turn, have the form, for each t:

(9) (a) $z_i \geq 0$

(b) $z_1 + z_2 + z_4 + z_5 + z_6 \leq x_3$

(c) $\epsilon_2 z_2 + \epsilon_4 z_4 + \epsilon_6 z_6 \leq x_5$

(d) $z_1 \leq f_2 x_2$

(e) $z_5 \leq f_6 x_6$.

We must now choose the $z_1(t)$ for $t = 0,1,2,\ldots,T-1$, subject to the above constraints, so as to maximize $x_1(T)$.

§3. Discussion.

As formulated, the problem above lies within the domain of linear programming. The problem of maximizing $x_1(T)$ may be solved for any given set of constants by a straightforward iterative process of the type developed by G. Dantzig and others.

A rapid count of the number of variables involved for T of moderate size, say 30, will show that even this simple three-industry problem with lumped capacities and stockpiles leads to a problem of huge dimensions. If we wish to determine the form of the solution corresponding to different initial states and corresponding to different sets of constants of proportionality, we are faced with a numerical task of vast magnitude. This type of analysis of the solution, a "sensitivity analysis," is required whenever we make the crude assumptions of lumping, linearity and so forth made above.

The question then arises as to whether or not it is possible to determine the intrinsic structure of an optimal policy, regardless of any numerical values which we may subsequently assign. This knowledge is not only of importante in itself, but is also of the utmost importance in determining approximate solutions in cases where any explicit analysis is too difficult.

For further discussion of this point we refer to [3], [4].

§4. A Continuous Version.

It is well-known in the physical world that continuous approximations are more amenable to mathematical analysis than discrete approximations, that calculus is a more powerful tool than algebra. Let us follow this lead and derive a continuous version of the equations in (2.8) and (2.9).

In passing, let us note that it may very well be that the continuous version we discuss may correspond more closely to the actual economic problem than the discrete versions which have heretofore been treated.

In the continuous version of the problem the allocations $z_i(t)$, over the time interval $[t,t+1]$ are replaced by the allocations $z_i(t)$ t over the time interval $[t,t+\Delta t]$. The quantities $z_i(t)$ are now rates of allocation. Writing out the equations corresponding to (8) and letting $\Delta t \geq 0$, the new equations take the form

(1)
$$\begin{aligned}
\dot{x}_1(t) &= \alpha_1 z_1(t) \\
\dot{x}_2(t) &= \alpha_2 z_2(t) \\
\dot{x}_3(t) &= -z_1(t)-z_3(t)-z_4(t)-z_5(t)-z_6(t) \\
\dot{x}_4(t) &= \alpha_4 z_4(t) \\
\dot{x}_5(t) &= -\epsilon_2 z_2(t)-\epsilon_4 z_4(t)-\epsilon_6 z_6(t)+\alpha_5 z_5(t) \\
\dot{x}_6(t) &= \alpha_6 z_6(t)
\end{aligned}$$

The constraints upon the z_i are now

(2)
$$\begin{aligned}
&(a) \quad z_i \geq 0 \\
&(b) \quad z_1+z_2+z_4+z_5+z_6 \leq \infty \\
&(c) \quad \epsilon_2 z_2+\epsilon_4 z_4+\epsilon_6 z_6 \leq \infty \\
&(d) \quad z_1 \leq f_2 x_2 \\
&(e) \quad z_5 \leq f_6 x_6
\end{aligned}$$

This means that the constraints of (2b) and (2c) disappear and are replaced by

(3) (b') $x_3 \geq 0$,

(c') $x_5 \geq 0$,

inequalities which were obviously satisfied previously.

The importance of these new constraints resides in the fact that when $x_3 = 0$, we must have

$$(4) \qquad z_1 + z_3 + z_4 + z_5 + z_6 \leq \gamma_2 x_4$$

and similarly when $x_5 = 0$ we must have

$$(5) \qquad \epsilon_2 z_2 + \epsilon_4 z_4 + \epsilon_6 z_6 \leq \alpha_2 z_5 \ .$$

Subject to the above equations constraints we wish to maximize $x_1(T)$.

§5. Some Remarks on Notation.

Before presenting our general theoretical approach, let us introduce a small amount of vector-matrix notation. Let $x(t)$ and $z(t)$ denote n-dimensional column vectors

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix}$$

and A_i, B_j, for such values of i and j as occur, denote n x m matrices.

By a non-negative vector $x(t)$ we shall mean one for which all of the components are non-negative, i.e., $x_i \geq 0$, and write $x \geq 0$. The inequality $x \geq y$ is to be equivalent to $x-y$ non-negative.

In terms of this notation the above equations and constraints have the form

(2) (a) $\frac{dx}{dt} = A_1 x + A_2 z$, $x(0) = c$,

(b) $z \geq 0$

(c) $B_1 z \leq B_2 z$.

To express our criterion function in simple form, let us introduce the vector inner product

(3) (a) $(x,y) = \sum_{i=1}^{n} x_i y_i$.

We may then write

(4) $x_1(T) = (x(T),\alpha)$,

where the first component of α is 1 and the remaining components are 0.

Finally, we shall write

(5) $\operatorname{Sup}_z (x(T),\alpha) = f(c,T)$,

where f is a scalar function of the vector c and the time variable T, i.e.,

(6) $f(c,T) \equiv f(c_1,c_2,\ldots,c_n;T)$.

Observe that we use Sup in (5) since the maximum may not be attained. We shall discuss this point again below.

§6. The Basic Functional Equation.

In writing (5.5) we have tacitly made use of the extremely important fact that the optimal yield, $x_1(T)$, is a function only of the initial state and the duration of the process, once the laws governing the process have been codified.

Let us further observe that the nature of an optimal allocation policy is such that its continuation over a sub-interval, say $[S,T]$, must be an optimal policy for a process of duration $T-S$ starting from the initial state $c(S)$.

The mathematical transliteration of this truism will yield the basic functional equation, namely

$$(1) \qquad f(c,S+T) = \operatorname*{Max}_{[0,S]} f(c(S),T) \; ,$$

where by $\operatorname*{Max}_{[0,S]}$ we mean maximization over all $z(t)$ for $0 \leq t \leq S$, satisfying the constraints. Let us for simplicity assume at the moment that the maximum is attained.

§7. Infinitesimal Analogue.

Let us now assume that $c(S) = x(S)$ possesses piecewise continuous derivatives and that f has the same property. Then taking $S = 0$ to be a well-behaved point, we have,

$$(1) \qquad x(S) = c + [A_1 c + A_2 z(0)]S + O(S) \; ,$$

and

$$(2) \qquad f(c,\, S+T) = f(c,T) + S\,\frac{\partial f}{\partial T} + O(S)$$

$$F(c(S),T) = f(c + [A_1 c + A_2 z(0)]S,T)$$

$$= f(c,T) + S(A_1 c + A_2 z(0),\, \frac{\partial f}{\partial c}\,) + O(S),$$

where

(3) $$\frac{\partial f}{\partial c} = \begin{pmatrix} \frac{\partial f}{\partial c_1} \\ \vdots \\ \frac{\partial f}{\partial c_n} \end{pmatrix}$$

Substituting in (6.1) and letting $S \to 0$, we see that infinitesimal generator of (1) is

(4) $$\frac{\partial f}{\partial T} = \operatorname*{Max}_{z(0)} \left[\left(A_1 c + A_2 z(0), \frac{\partial f}{\partial c} \right) \right] ,$$

where the maximum is now taken over an n-dimensional z-region determined by the constraints.

The importance of the infinitesimal analogue is that it permits us to determine the solution over $[0, T+\Delta T]$ if we have it determined over $[0,T]$ for <u>all</u> initial states. Since in most of these problems the difficulties are readily resolved for small T, we have a feasible approach.

As an illustration of these remarks we shall plough through the details of a relatively simple problem below.

§8. A Particular Problem.

Let us now consider the problem of maximizing $x_2(T)$ given the following equations and constraints

(1) $$\frac{dx_1}{dt} = a_1 t_1, \quad x_1(0) = c_1 ,$$

$$\frac{dx_2}{dt} = a_2 t_2 - t_1, \quad x_2(0) = c_2 ,$$

where z_1 and z_2 are subject to the constraints

(2) (a) $z_1, z_2 \geq 0$

(b) $z_1 + z_2 \leq x_2$

(c) $z_2 \leq x_1$

(d) $x_2 \geq 0$.

Let

(3) $$\underset{[0,T]}{\text{Max}}\ x_2(T) = f(c_1, c_2, T).$$

It is easy to see, using either classical theorems in the calculus of variations, or a weak convergence argument, that the maximum is assumed in this case.

As in §6, f satisfies the functional equation

(4) $$f(c_1, c_2, S+T) = \underset{[0,S]}{\text{Max}}\ f(x_1(S), x_2(S), T),$$

which leads to as in §7, the non-linear partial differential equation

(5) $$\frac{\partial f}{\partial T} = \underset{z(0)}{\text{Max}} \left[a_1 z_1 \frac{\partial f}{\partial c_1} + (a_2 z_2 - z_1) \frac{\partial f}{\partial c_2} \right].$$

The maximum is taken over the region defined by

(6) (a) $0 \leq z_1, z_2$

(b) $z_1 + z_2 \leq c_2$

(c) $z_2 \leq c_1$,

with the additional restraint

(7) $a_2 z_2 - z_1 \geq 0$

if $x_2 = 0$.

Let us now sketch the analytic procedure that will yield a solution. We begin with the most complicated case, that where $c_2 < c_1$.. For a process of short duration, the solution is trivial. We have

$$(8) \qquad z_1 = 0, \qquad z_2 = x_2 , \qquad f = c_2 e^{a_2 T} .$$

This policy is pursued until a "bottleneck" develops, which is to say c_2 exceeds c_1. Using the optimal policy described in (8) we see that this situation will occur as soon as T exceeds $T_1 = \log (c_1/c_2)/a_2$.

To obtain the solution for $T > T_1$, we rewrite (5) in the form

$$(9) \quad \frac{\partial f}{\partial T} = \operatorname*{Max}_{z(0)} \left[z_1 \left(a_1 \frac{\partial f}{\partial c_1} - \frac{\partial f}{\partial c_2} \right) + a_2 \frac{\partial f}{\partial c_2} z_2 \right] .$$

The location of the maximizing point $(z_1(0), z_2(0))$ will depend upon the sign and magnitude of the coefficients of g_1 and g_2. For $T < T_1$ we have

$$(10) \quad a_1 \frac{\partial f}{\partial c_1} - \frac{\partial f}{\partial c_2} = - e^{a_2 T} , \qquad a_2 \frac{\partial f}{\partial c_2} = a_2 e^{a_2 T} .$$

Using our assumption concerning the continuity of $\frac{\partial f}{\partial c_1}$, $\frac{\partial f}{\partial c_2}$ we suspect that the solution for T slightly greater than T_1 will have the form

$$(11) \quad (a) \quad z_1 = 0, \ z_2 = x_2 \quad \text{for} \quad 0 \le S \le T_1$$
$$z_1 = 0, \ z_2 = x_1 \quad \text{for} \quad T_1 < S \le T .$$

Applying this policy, f takes the form

(12) $$f = b_2c_1 + (T-T_1)\, a_2b_2c_1 ,$$

where T_1 is as above. In order to determine how long this policy endures when $T > T_1$, we consider the process as starting from $S = T_1$. In terms of $c_1' = c_1(T_1)$, $c_2' = c_2(T_1)$, f has the form

(13) $$f = c_2' + a_2c_1'(t-T_1).$$

The equation which replaces (9) has precisely the same form with c_1, c_2 replaced by c_1', c_2', namely

(14) $$\frac{\partial f}{\partial T} = \operatorname*{Max}_{z(T_1)} \left[z_1\left(a_1 \frac{\partial f}{\partial c_1'} - \frac{\partial f}{\partial c_2'}\right) + a_2 \frac{\partial f}{\partial c_2'} z_2 \right] .$$

We have, using (13),

(15) $$a_1 \frac{\partial f}{\partial c_1'} - \frac{\partial f}{\partial c_2'} = a_1a_2(T-T_1) - 1,$$
$$a_1 \frac{\partial f}{\partial c_1'} = a_2 .$$

The coefficient of z_1 is negative for $T < T^* = T_1 + 1/a_1a_2$, 0 at T^*, and positive thereafter.

It follows that the new policy given by (11) remains optimal for $T_1 \le T \le T^*$.

Furthermore since T^*-T_1 is independent of c_1 and c_2 we see that we know the form of the optimal policy over a tail interval.

It remains to determine what the policy is in the middle in the general case when T exceeds T_1^*. We suspect that it has the form

(16) $$z_1 = x_2 - x_1 , \quad z_2 = x_1 .$$

In place of verifying this directly, which may be done, we shall describe in the next section a more elegant technique which exploits the linearity of the equations.

§9. A Dual Problem.

Let us take our basic equations to have the form

(1) $$\frac{dx}{dt} = Az, \quad z(0) = c,$$

with constraints of the form

(2) (a) $z \geq 0$

(b) $Bz \leq x$.

Since $x = c + \int_0^t Azdt$, the constraint may be written

(3) $$Bz + \int_0^t Czdt \leq c, \qquad (C = -A) .$$

The problem of maximizing $(x(T),\alpha)$ is equivalent to that of maximizing $\int_0^T (Az,\alpha)dt = \int_0^T (z,\alpha')dt$.

Beginning all over again, we start with the problem of maximizing $J = \int_0^T (z,\alpha')dt$ over all z satisfying the constraints

(4) (a) $z \geq 0$

(b) $Bz + \int_0^T Czdt_1)dt \leq c$

Let w(t) be a non-negative vector of the same dimension as c. Then by virtue of (4b), we have

(5) $$\int_0^T (w,Bz + \int_0^t Czdt_1)dt \leq \int_0^T (w,c)dt.$$

Let B' denote the transpose of B. Then, as is easily seen, $(Bz,w) = (z,B'w)$. Integration by parts yields, for any constant matrix C,

$$(6)\qquad \int_0^T (w, \int_0^t Czdt_1)dt = \int_0^T (\int_t^T C'\, wdt, z)dt.$$

Combining these two results, we have

$$(7)\qquad \int_0^T (w, Bz + \int_0^t Czdt_1)dt = \int_0^T (B'w + \int_t^T C'wdt, z)dt$$

Let us now assume that it is possible to find a vector $w = w(t)$ which is non-negative and satisfies the inequality

$$(8)\qquad B'w + \int_t^T C'wdt_1 \geq \alpha'$$

We then have the chain of equalities and inequalities

$$(9)\qquad \int_0^T (w,c)dt \geq \int_0^T (w, Bz + \int_0^t Czdt_1)dt =$$

$$= \int_0^T (B'w + \int_t^T C'wdt, z)dt \geq \int_0^T (\alpha', z)dt .$$

From this it is clear that

$$(10)\qquad \operatorname*{Inf}_w \int_0^T (w,c)dt \geq \operatorname*{Sup}_z \int_0^T (z,\alpha')dt ,$$

where the infimum and supremum are taken over all w and z satisfying the inequalities of (8) and (4b). If the minimum and maximum are assumed, the details are as above. If, however, the minimum and maximum are not assumed, then delta-functions will occur, which is to say, we must reformulate the problems in terms of Stieltjes integrals. A number of interesting and difficult problems arise in this way, which we shall not discuss here.

If the two extremes in (10) are equal, we see that the following relations must hold,*

$$(11)\qquad \begin{aligned} w_i &= 0 \quad \text{if} \quad c_i > \left(Bz + \int_0^t Czdt\right)_i \\ z_j &= 0 \quad \text{if} \quad \alpha'_j < \left(B'w + \int_t^T C'wdt\right)_j \end{aligned}$$

The important fact which we now wish to establish is that, conversely, any pair of non-negative z and w satisfying (11) and the original constraints will furnish solutions to the maximum and minimum problems.

To demonstrate this, let us note that if (12) holds, all the relations in (9) are equalities. Assume now that $\bar{z}$ is another vector satisfying all the constraints and for which

$$(12)\qquad \int_0^T (z,\alpha')dt < \int_0^T (\bar{z},\alpha')dt \ .$$

Then with the w associated with z we have

$$(13)\qquad \int_0^T (\bar{z},\alpha')dt \le \int_0^T \left(\bar{z},\ B'w + \int_t^T A'wdt_1\right)dt =$$

$$\int_0^T \left(B\bar{z} + \int_0^t A\bar{z}dt_1, w\right)dt \le \int_0^T (c,w)dt =$$

$$\int_0^T (z,\alpha')dt,$$

a contradiction.

* Apart from sets of measure zero, to be technical.

It follows then that we have a systematic procedure for verifying a conjectured solution. Given z, we determine w by means of (11), and then see whether or not w satisfies the given constraints. In the next section we shall carry through the details for the problem of §8.

§10. Verification of the Solution Given in §8.

Applying the techniques described above, we find that the dual of the problem proved in §8 is that of minimizing $\int_0^T (c_1 w_1 + c_2 w_2)dt$, where

$$\text{(1)} \qquad \frac{dy_1}{dt} = -a_1 w_1 + w_2, \quad y_1(T) = -1 \,,$$

$$\frac{dy_2}{dt} = -a_2 w_2, \qquad y_2(T) = a_2$$

and the constraints have the form

(2) (a) $w_1, w_2 \geq 0$

(b) $w_1 + w_2 \geq y_2$

(c) $w_2 \geq y_1$

The equations of (9.11) are now:

If

(3) (a) $z_2 < x_1$, then $w_1 = 0$

(b) $z_1 + z_2 < x_2$, then $w_2 = 0$

(c) $w_2 > y_1$, then $z_1 = 0$

(d) $w_1 + w_2 > y_2$, then $z_2 = 0$.

We have omitted the conditions corresponding to $x_2 \geq 0$ since we suspect that the optimal allocation policy automatically keeps $x_2 > 0$.

We wish to verify that the policy which maximizes $x(T)$ is

(4) (a) For $T-1/a_1a_2 < t \leq T$, $z_1(t) = 0$, $z_2 = \text{Min}\,(x_1, x_2)$

(b) For $0 \leq t \leq T-1/a_1a_2$, (1) if $x_2 \leq x_1$, $z_1 = 0$, $z_2 = x_2$

(2) if $x_2 \geq x_1$, $z_1 = x_2 - x_1$

$z_2 = x_1$.

It is easily seen that this is a permissible policy in that $z_1 = x_2 - x_1$ is actually non-negative when z_1 and z_2 have the above values.

Having prescribed z, we can determine w using (3) and then test for consistency. There are two cases to consider, depending upon whether x_2 ever exceeds x_1 or not.

Let us assume then that $T \geq T_1$, in which case x_2 can exceed x_1 if appropriate policies are used.

<u>Case I</u>: $T - 1/a_1a_2 < T_1 < T$. The solution is given by

(5) for $t < T_0$: $z_1 = 0$, $z_2 = x_2$

for $t \geq T_0$: $z_1 = 0$, $z_2 = x_1$.

For $t < T_1$ these results yield, in conjunction with (3),

(6) for $t < T_0$, $w_1(t) = 0$, $w_2(t) = y_2(t)$

for $t \geq T_0$, $w_2(t) = 0$, $w_1(t) = y_2(t)$.

For $t > T_0$ we obtain, using (1)

(7) $y_2(t) = a_2$, $y_1(t) = -1 + a_1a_2(T-t) < 0$,

while for $t < T_1$ we have

(8) $y_2(t) = a_2 e^{a_2(T_1-t)} > 0$,

$y_1(t) = -1 + a_1 a_2(T-T_1) - e^{a_2(T_1-t)} < 0$

Hence, the inequalities $w_1, w_2 \geq 0$, $w_2 \geq y_1$, $w_1 \geq y_2$ are satisfied in their respective intervals.

Case II. $T_1 < T - 1/a_1 a_2$. This is the most interesting case. The vectors z and w are now determined as follows:

(9) for $T - 1/a_1 a_2 \leq t \leq T$: $z_1 = 0$ $w_2 = 0$

$z_2 = x_1$, $w_1 = y_2$

for $T_0 \leq t \leq T - 1/a_1 a_2$: $z_1 = x_2 - x_1$ $w_2 = y_1$

$z_2 = x_1$, $w_1 = y_2 - y_1$

for $0 \leq t \leq T_0$: $z_1 = 0$ $w_1 = 0$

$z_2 = x_2$ $w_2 = y_2$

For $T - 1/a_1 a_2 \leq t < T$ we have

(10) $y_2(t) = a_2$, $y_1(t) = -1 + a_1 a_2(T-t)$.

Hence, in this interval $y_1(t) \leq 0 = w_2$. Note that $y_1(T-1/a_1 a_2) = 0$. In the range $T_0 < t \leq T - 1/a_1 a_2$, we have the equations

(11) $$\frac{dy_1}{dt} = -a_2 y_2 + (a_1+1) y_1$$

$$\frac{dy_2}{dt} = -a_2 y_2 .$$

Let us show that $y_1 \geq 0$ and $y_2 \geq y_1$ in this range. Starting from $t = T - 1/a_1 a_2$ where the inequalities are satisfied, let us reverse the time. The backward equations are

(12) $$\frac{dy_1}{dt} = a_2y_2 - (1+a_1)y_1$$

$$\frac{dy_2}{dt} = a_2y_2$$

From this we obtain

(13) $$\frac{d}{dt}(y_2-y_1) = (1+a_1)y_1 \ .$$

Hence, if y_1 remains non-negative, we will have $y_2-y_1 \geq 0$. It is clear that dy_1/dt starts out positive and stays positive as long as (y_1,y_2) remains above $a_2y_2 - (1+a_1)y_1 = 0$. If it hits the line we have $dy_1/dt = 0$, which means that y_1 has a maximum or a point of inflection. Both are excluded, since

(14) $$\frac{d^2y_1}{dt^2} = a_2\frac{dy_2}{dt} - (1+a_1)\frac{dy_1}{dt} = a_2^2\, y_2 > 0.$$

This shows that w_1 and w_2 remain non-negative in this interval.

Finally for $t < T_o$ we have

(15) $$\frac{dy_1}{dt} = y_2, \ \frac{dy_1}{dt} = -a_2y_2 \ .$$

As t decreases, y_2 increases and y_1 decreases. Hence, $y_2 > y_1$ remains valid.

This completes the verification.

§11. Non-Linear Problems.

A great number of problems in mathematical economics reduce to the maximization of an integral of the form

(1) $$T = \int_o^T F(x_1,x_2,\ldots,x_n,\ z_1,z_2,\ldots,z_n)dt$$

and there a number of constraints of the form

(3) $$R_k(x_1,x_2,\dots,x_n, z_1,z_2,\dots,z_n) \leq 0 ,$$

cf. [ ], [ ] .

These problems may also be approached by the functional equation outlined above. A brief outline of the procedure, together with an extension to eigenvalue problems, will be found in [ ] .

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